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Buoyancy effects upon lateral dispersion in open-channel flow

By RONALD SMITH
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

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A model equation is derived for the combined longitudinal and lateral dispersion of a buoyant contaminant in open-channel flow. The central hypotheses are that the water is shallow and that to a first approximation the effluent is vertically well mixed. The model includes allowance for the reduction in turbulent intensity due to weak vertical stratification, a buoyancy-driven secondary flow, and the redistribution of longitudinal momentum by the secondary flow. For a plume the theoretical results for the excess spreading due to buoyancy agree well with Prych's (1970) experimental results.

1. Introduction

In studies of thermal and of sewage effluent discharges it is natural to regard the flow field as being comprised of several zones with distinctive properties, particularly in terms of the consequences of the discharge being buoyant. For an outlet in a shallow river or estuary a typical division would be: a near field with predominant influence of the outfall geometry and for which vertical stratification can inhibit vertical mixing (figure 1); a middle field with lateral mixing enhanced by a buoyancy-driven secondary flow (figure 2); and a far field in which the contaminant distribution has become almost uniform across the channel, though there may still be sufficient lateral density variation to drive a significant secondary flow and there can also be changes in the longitudinal current which can be interpreted as being part of a horizontal circulation (figure 3). In recent years there has been considerable progress in the development and verification of mathematical models appropriate to each of these zones (e.g. Lee, Jirka & Hartleman 1974; Prych 1970; Imberger 1976). However, the simplifying assumptions used in the derivations of the model equations are mutually exclusive (i.e. the neglect of the dominant features of the adjacent zones). Thus problems remain as to the transition between the zones (Abraham 1976). One possible means of progress is to relax the assumptions and thereby extend the range of validity of the separate models.

Here the model which we seek to extend is that due to Prych (1970) for lateral mixing of a buoyant contaminant in open-channel flow (i.e. for the middle field). Specifically, we assume that in the absence of buoyancy there is steady unidirectional flow in a wide channel of constant depth. Moreover, we shall restrict our attention to regions of the flow in which the effluent has become vertically well mixed.

In a critical assessment of his model equation, Prych (1970) adjudged the main shortcoming to be the neglect of the effects of longitudinal velocity gradients and of

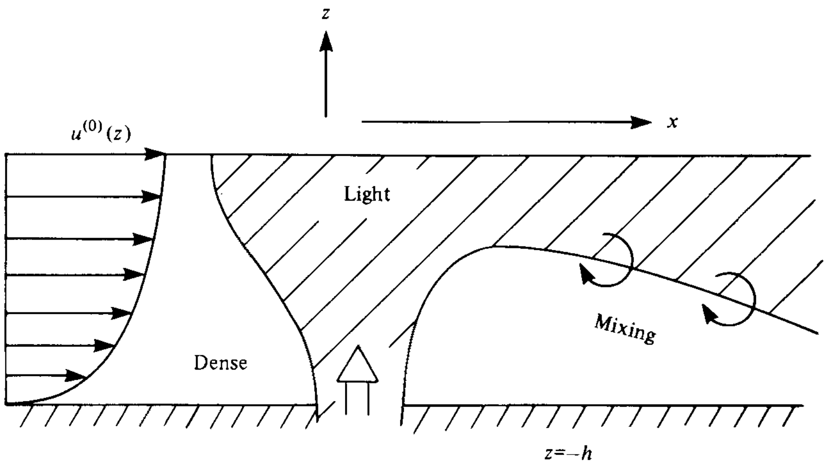

FIGURE 1. Longitudinal section illustrating the near field.

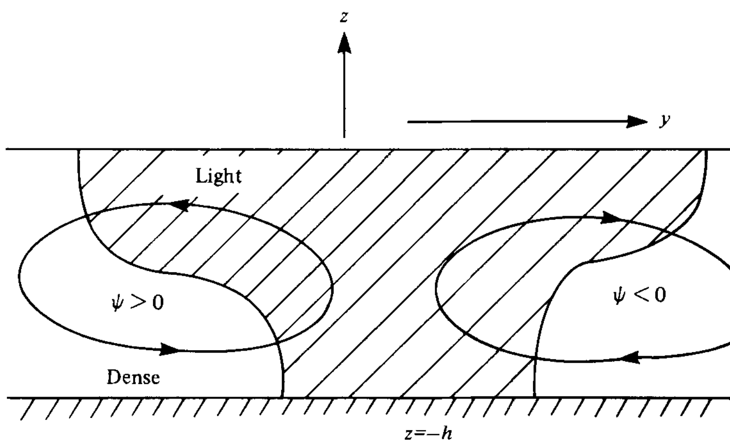

FIGURE 2. Cross-section illustrating the secondary flow.

vertical stratification upon the secondary flow. He argued that the longitudinal current carried with it lateral momentum from upstream regions of strong secondary flow to downstream regions of weaker flow. Also, the secondary flow induces weak vertical stratification (see figure 2), which reduces the turbulent intensity. Both these near-field effects tend to increase the secondary flow and hence to increase the width of a plume. This would help to explain the fact that Prych's theory consistently underestimated the excess spreading due to buoyancy.

The original intention of the present work was to take up these suggestions and to extend the validity of Prych's middle-field model back towards the near field. However, it turns out that there are closely related, physical effects which continue to be important further into the middle field other than those suggested by Prych. In the outer part of a buoyant plume the secondary flow carries with it longitudinal momentum from the free surface down towards the channel bed. Also, throughout the plume the stratification reduces the turbulent intensity and therefore reduces the local flow resistance to the downstream pressure gradient. These two effects tend

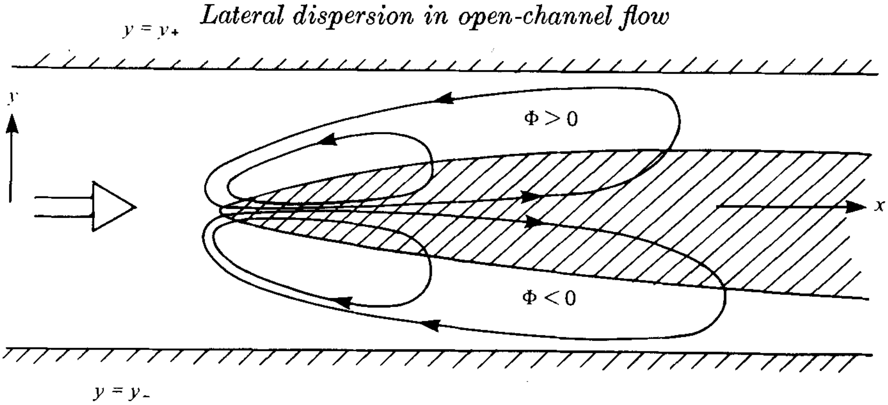

FIGURE 3. Plan view illustrating the horizontal circulation.

to increase the longitudinal current locally. The strength of this excess current decreases downstream and, in order to preserve the total volume of water, there must be a horizontal circulation as sketched in figure 3. Locally this circulation is outward and so we again predict an increased width for a buoyant plume.

The mathematical analysis, which leads to the above identification of the dominant physical mechanisms, is based upon the use of maximum-generality scalings. That is, we identify a regime, for the geometrical and flow parameters, in which the features retained by Prych (1970), and as many as possible additional features, have a leading-order effect upon the evolution of the contaminant distribution. In adjacent parameter regimes the physics are simpler but are correctly described by the model equation, the negligible physical effects merely corresponding to numerically insignificant terms. By construction, the union of the adjacent parameter regimes is the maximal extension of the range of validity of a middle-field model.

The model equation thus obtained takes the form

$$\partial_t(h\|c\|) + \partial_x(h\|u^{(0)}\|\,\|c\|) + \partial_y\,\Phi\partial_x\|c\| - \partial_x\,\Phi\partial_y\|c\| = (\partial_x, \partial_y)\begin{bmatrix} E & D_1 \\ D_1 & K \end{bmatrix}\begin{bmatrix} h\,\partial_x\|c\| \\ h\,\partial_y\|c\| \end{bmatrix}. \quad (1)$$

Here x and y are distances in the longitudinal and lateral directions, h is the (constant) water depth, $\|c\|$ the vertically averaged concentration, $\|u^{(0)}\|$ the (constant) bulk velocity in the absence of the effluent, Φ a stream function for the buoyancy-driven horizontal circulation, and E, D_1 and K are dispersion coefficients. The presence of the off-diagonal term D_1 in the dispersion matrix simply means that the directions for maximum and minimum dispersion are not aligned along and across the channel. This is due to there being vertical shear associated with both the primary current and the secondary flow (see figure 4).

If the undisturbed velocity profile is logarithmic with friction velocity u_*, then the coefficients in the model equation (1) are found to be

$$E = 6{\cdot}3hu_*, \quad D_1 = 0{\cdot}98h^3\alpha g\,\partial_y\|c\|/u_*, \quad (2a, b)$$

$$K = 0{\cdot}15hu_* + 0{\cdot}16h^5(\alpha g\,\partial_y\|c\|)^2/u_*^3, \quad (2c)$$

$$\Phi = -2{\cdot}1\int_{y-}^{y}\left[\frac{h^3}{u_*}\alpha g\,\partial_x\|c\| - \overline{\frac{h^3}{u_*}\alpha g\,\partial_x\|c\|}\right]dy + 0{\cdot}70\int_{-y}^{y}\left[\frac{h^5}{u_*^3}(\alpha g\,\partial_y\|c\|)^2 - \overline{\frac{h^5}{u_*^3}(\alpha g\,\partial_y\|c\|)^2}\right]dy + 0{\cdot}78h^4\|u^{(0)}\|\,\alpha g\,\partial_y\|c\|/u_*^2. \quad (2d)$$

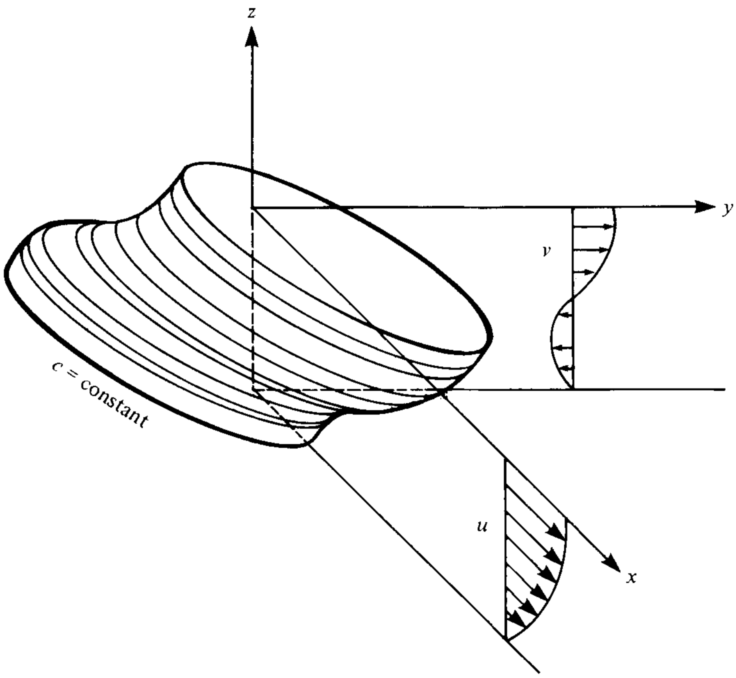

FIGURE 4. Perspective view of a concentration contour when there is shear flow in two directions.

Here αg is the reduced gravity (positive for a buoyant contaminant), $y = y_-$ denotes the left-hand side of the channel (viewed from upstream) and the overbars indicate cross-sectional average values. For an extremely wide channel we can set $y_- = -\infty$ and neglect the averaged terms.

In the limit of no y dependence (i.e. a very wide contaminant distribution) (1) becomes

$$\partial_t \|c\| + \|u^{(0)}\| \, \partial_x \|c\| = \partial_x (E \, \partial_x \|c\|),$$

in agreement with the work of Elder (1959). Similarly, if we first convert to axes moving with the bulk velocity and then neglect all x derivatives, (1) takes the form

$$\partial_t \|c\| = \partial_y ([D_0 + (\alpha g \, \partial_y \|c\|)^2 D_2] \, \partial_y \|c\|). \qquad (3)$$

This is the Erdogan–Chatwin (1967) equation, which was derived in the present physical context by Prych (1970). However, the nonlinear coefficient D_2 given by (2c) is over three times as large as that calculated by Prych for laminar flows. This change alone goes a long way towards explaining the disparity between Prych's (laminar) theory and his (turbulent) experiments.

The new equation (1) has terms which are not present in either of the two previously studied limits. As has been noted above, D_1 is related to the vertical shear in both the longitudinal current and the buoyancy-driven secondary flow. The two integral terms in the expression (2d) for the horizontal circulation show the effect of the reduced turbulent intensity, while the derivative term is associated with the redistribution of longitudinal momentum by the secondary flow.

Approximations to the full model equation are sought, first on the basis of the fact that the friction velocity u_* is commonly very small relative to the bulk velocity $\|u^{(0)}\|$, then, for the particular case of a steady plume in an extremely wide channel, on the basis of the further assumption that the contaminant distribution is nearly Gaussian. An ordinary differential equation is obtained for the standard deviation of a plume. For strongly buoyant narrow plumes, with predicted and measured Richardson numbers in excess of 0·25, the theory underestimates the dispersion. Presumably this is attributable to additional physical mechanisms, such as lateral gravity currents spreading out across the free surface (or channel bed), which are not represented in the vertically averaged middle-field equations. However, the greater part of Prych's (1970) experiments are compatible with the hypothesis of efficient vertical mixing. For these wider, less buoyant plumes there is good agreement between theory and experiment.

2. Choice of scalings

As the starting point for our mathematical analysis we take the channel to be straight and of constant depth; turbulent transports are represented by eddy-diffusivity tensors with principal axes in the longitudinal, lateral and vertical directions; and we make the Boussinesq approximation, in which we include the buoyancy effect due to density variations but neglect the associated inertia variations. Thus, written in full, the equations of motion and boundary conditions are

$$\partial_t c + u\,\partial_x c + v\partial_y c + w\partial_z c = \partial_x(\kappa_1\partial_x c) + \partial_y(\kappa_2\partial_y c) + \partial_z(\kappa_3\partial_z c), \tag{4a}$$

$$\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u + \partial_x p - G = \partial_x(2\nu_{11}\partial_x u) + \partial_y[\nu_{12}(\partial_y u + \partial_x v)] + \partial_z[\nu_{13}(\partial_z u + \partial_x w)], \tag{4b}$$

$$\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + \partial_y p = \partial_x[\nu_{12}(\partial_y u + \partial_x v)] + \partial_y(2\nu_{22}\partial_y v) + \partial_z[\nu_{23}(\partial_z v + \partial_y w)], \tag{4c}$$

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \partial_z p - \alpha g c = \partial_x[\nu_{13}(\partial_z u + \partial_x w)] + \partial_y[\nu_{23}(\partial_z v + \partial_y w)] + \partial_z(2\nu_{33}\partial_z w), \tag{4d}$$

$$\partial_x u + \partial_y v + \partial_z w = 0, \tag{4e}$$

$$u = v = w = \kappa_3\partial_z c = 0 \quad \text{on} \quad z = -h, \tag{4f}$$

$$\nu_{12}\partial_z u = \nu_{23}\partial_z v = w = \kappa_3\partial_z c = 0 \quad \text{on} \quad z = 0, \tag{4g}$$

$$Ri = \alpha g\,\partial_z c/[(\partial_z u)^2 + (\partial_z v)^2] \quad \text{with} \quad \kappa_i = \kappa_i(Ri), \quad \nu_{ij} = \nu_{ij}(Ri). \tag{4h}$$

Here (x, y, z) are distances in the longitudinal, lateral and vertical directions, (u, v, w) are the corresponding velocity components, κ_i and ν_{ij} are the eddy diffusivities for concentration and momentum, G is the pressure gradient which drives the basic flow, p the pressure perturbation, h the constant water depth, c the concentration, αg the reduced gravity (positive for a buoyant contaminant) and Ri the gradient Richardson number.

Our objective is to achieve the simplifications of (4) implicit in the concept of the middle field, with a minimum loss of generality. Physically, the main feature that distinguishes the middle field from the near field is that the effluent distribution has become vertically well mixed. This is not a condition which can readily be used as the basis of a mathematical analysis. Instead, we follow Prych (1970) and impose the requirement that the lateral scale B of the concentration distribution greatly exceeds the water depth scale H, i.e. that the ratio

$$\delta = H/B \tag{5}$$

be small. To infer the absence of strong vertical concentration gradients we need the further mild restriction that the lateral and vertical turbulent diffusion coefficients are of the same order. It follows that, in the time necessary for the effluent discharge to achieve the width B, the vertical concentration variation will have become of order δ or less. Indeed, if it were not for vertical non-uniformities of the current, the vertical concentration gradient would be exponentially small.

An immediate implication of the efficient vertical mixing is that to the first approximation the effluent is carried along at the vertically averaged velocity $\|u\|$. This basic state involves no dispersion. Thus, as was first explicitly recognized by Taylor (1953), the dispersion depends upon various small effects such as the residual vertical concentration gradient. Here we hope to include the transverse turbulent diffusion, a density-induced secondary flow and as many as possible additional effects. The mathematical device which facilitates the achievement of this aim is the precise specification of the δ-ordering of the many terms with respect to the basic dimensional quantities H and $\|u\|$.

The original method by which the author derived the parameter regime of interest was by commencing with arbitrary scalings and then assuming that more and more physical effects have a leading-order effect upon the evolution of the contaminant distribution until the scalings became fully determined. The same results can be derived more systematically by following through the somewhat lengthy argument symbolized in figure 5.

Suppose that the turbulent diffusivities are of order $H\|u\|\delta^{\beta}$, where β is to be determined. The secondary flow augments the lateral turbulent diffusion by a (lateral) shear dispersion coefficient with the dimensional form

$$H^2\|v^2\|/\kappa_3 \sim \delta^{-\beta} H\|v^2\|/\|u\|$$

(Taylor 1953), where v is the lateral velocity and κ_3 the vertical diffusivity for the contaminant. For this shear dispersion to have an effect upon the evolution of the contaminant distribution comparable to that of the turbulent field lateral diffusivity κ_2, it is necessary that the lateral velocity scale for the density-driven secondary flow be of order $\|u\|\delta^{\beta}$. Next, we assess the density variations necessary to maintain this secondary flow. From the lateral momentum equation we infer that $\partial_y p$ is of order $\delta^{2\beta}\|u\|^2/H$ and hence that the pressure perturbation p is of order $\delta^{2\beta-1}\|u\|^2$. A hydrostatic approximation to the vertical momentum equation (4d) reveals that this pressure is associated with a contaminant concentration of order unity provided that the reduced gravity αg is of order $\delta^{2\beta-1}\|u\|^2/H$. From the diffusion equation (4a) we find that the vertical non-uniformity of the secondary flow leads to a vertical concentration gradient of order δ/H. Together the estimates of αg and $\partial_z c$ permit us to deduce that the Richardson number is of order $\delta^{2\beta}$. (Here we have made the mild

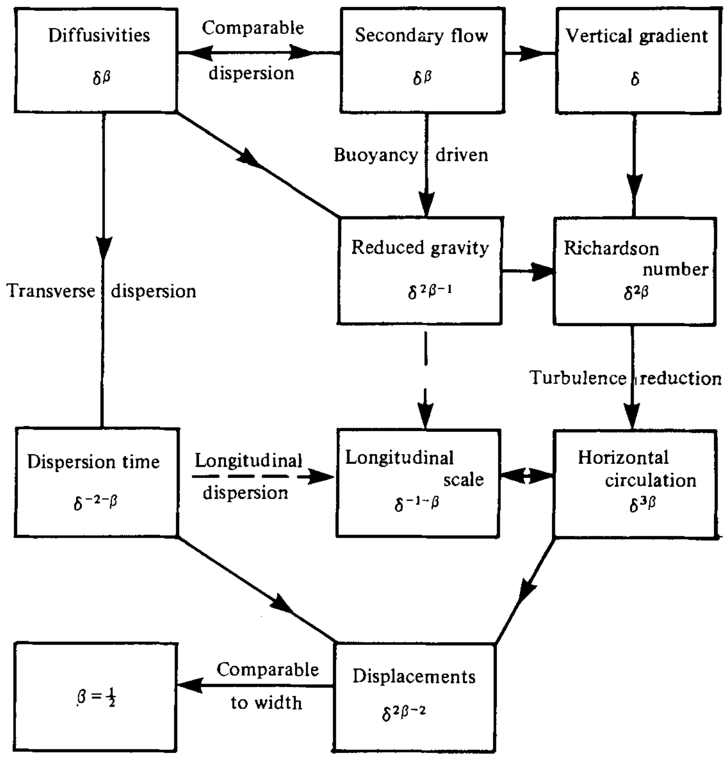

FIGURE 5. Derivation of the maximum-generality scalings.

assumption that $\beta \geqslant 0$ in order to neglect the $(\partial_z v)^2$ influence upon the value of *Ri*.) For weak stratification the reductions in the turbulent transports from their non-stratified values are known to be proportional to the Richardson number. Equivalently, the fractional increase in the longitudinal or lateral velocities for a given horizontal pressure gradient is proportional to *Ri*. The implied change in the lateral velocity is of order $\|u\|\delta^{3\beta}$. Similar small velocity changes are caused by $u\partial_x v$ convective derivatives in the lateral momentum equation (4*c*), provided that the longitudinal length scale is of order $H\delta^{-\beta-1}$. (It happens that this is precisely the length scale for which the two further effects of longitudinal shear dispersion and longitudinal density-driven currents become important.) Unlike the dominant (secondary flow) contribution to the lateral velocity, these additional (horizontal circulation) velocity perturbations need not have zero vertical averages. Thus there can be an associated lateral drift of the concentration distribution. With a lateral diffusivity of order $h\|u\|\delta^{\beta}$ it takes a time scale of order $\delta^{-\beta-2}H/\|u\|$ for the contaminant distribution to achieve a width of order $\delta^{-1}H$. In this length of time the drift velocities lead to lateral displacements of $H\delta^{2\beta-2}$. This is comparable to the width scale of the contaminant distribution provided that $\beta = \frac{1}{2}$. Having at last determined β, we can follow through the argument once more and evaluate the maximum-generality scalings for the numerous terms.

Summing up the results of either the original *ad hoc* or the above deductive argument (and ignoring the dimensional H and $\|u\|$ factors) we have the following: the eddy diffusivities and the lateral velocities are of order $\delta^{\frac{1}{2}}$; the reduced gravity is of order unity; the width is by definition of order δ^{-1}; the longitudinal extent of the contaminant distribution is of order $\delta^{-\frac{3}{2}}$; and the evolution time scale is of order $\delta^{-\frac{5}{2}}$. The importance of these scalings is that they represent the most complicated possibility with respect to the several physical mechanisms retained. Thus, for a reasonably wide parameter range we can assert that no significant terms are neglected though some numerically small terms may be retained.

It is important to check that the maximum-generality scalings do represent a physically realistic possibility, as otherwise the increased range of validity may be illusory. For a shallow river a feasible specification is

$$\|u\| = 0{\cdot}1 \text{ m s}^{-1}, \quad H = 2 \text{ m}, \quad B = 50 \text{ m}.$$

The implied scales for the other major physical parameters are

eddy diffusivities $\sim$ 0·04 $\text{m}^2\,\text{s}^{-1}$, length $\sim$ 250 m,
total weight deficit (or excess) due to the contaminant $\sim$ 12·5 tonnes.

These values, in particular those for the eddy diffusivities, are realistic. Of course, the precise matching of the mass deficit, length and breadth scales is exceptional. In general, as a contaminant cloud evolves different terms in the maximum-generality equation will be negligible.

In axes moving at velocity $\|u^{(0)}\|$, and with the relative sizes of terms made explicit, the equations of motion (4a–f) can be rewritten:

$$\delta^2\partial_t c + \delta(u - \|u^{(0)}\|)\,\partial_x c + \delta v\partial_y c + \delta w\partial_z c = \partial_z(\kappa_3\partial_z c) + \delta^2\partial_y(\kappa_2\partial_y c) + O(\delta^4), \tag{6a}$$

$$\delta(u - \|u^{(0)}\|)\,\partial_x u + \delta v\partial_y u + \delta w\partial_z u - G + \delta\partial_x p = \partial_z(\nu_{13}\partial_z u) + O(\delta^2), \tag{6b}$$

$$\delta(u - \|u^{(0)}\|)\,\partial_x v + \delta v\partial_y v + \delta w\partial_z v + \partial_y p = \partial_z(\nu_{23}\partial_z v) + O(\delta^2), \tag{6c}$$

$$\partial_z p - \alpha g c = O(\delta^2), \tag{6d}$$

$$\partial_x u + \partial_y v + \partial_z w = 0, \tag{6e}$$

$$u = v = w = \kappa_3\partial_z c = 0 \quad \text{on} \quad z = -h, \tag{6f}$$

$$\nu_{12}\partial_z u = \nu_{23}\partial_z v = w = \kappa_3\partial_z c = 0 \quad \text{on} \quad z = 0, \tag{6g}$$

$$Ri = \alpha g\partial_z c/[(\partial_z u)^2 + \delta(\partial_z v)^2] \quad \text{with} \quad \kappa_i = \kappa_i(Ri), \quad \nu_{ij} = \nu_{ij}(Ri). \tag{6h}$$

A nice feature of these non-dimensional equations is that we can revert to dimensional variables simply by setting $\delta = 1$ and re-interpreting the symbols as their dimensional counterparts.

Superficially there is a resemblance to the equations (1a–g) used by Smith (1976) in a study of the far field. This resemblance is primarily due to the similar physical context. It rapidly becomes apparent below that the differences in scalings for middle-field and far-field problems lead to mathematical changes commensurate with the physical changes.

3. Regular expansion

It is implicit in the above choice of scalings that all the flow variables have derivatives of order unity with respect to the scaled (non-dimensional) co-ordinates x, y, z and t. Thus any dependence upon the small parameter δ must be regular. The form of (6) reveals that the appropriate representation of the flow variables is

$$u = u^{(0)} + \delta u^{(1)} + \delta^2 u^{(2)} + \ldots,$$

where the $u^{(j)}$ are all independent of δ. We now follow through the consequences of this simple representation, which lead eventually to a single equation involving only $c^{(0)}$.

Since our interest is primarily in the contaminant dispersion, a central role in the calculations is played by the diffusion equation (6a) together with the zero-flux boundary conditions. At leading order in δ there is the trivial solution

$$c^{(0)}(x, y, t) \quad \text{independent of } z, \tag{7a}$$

and without loss of generality $c^{(0)}$ can be identified with the vertically averaged concentration $\|c\|$. At the next order the convective derivatives of c provide forcing terms. A necessary condition for a solution to exist is that these terms have zero vertical average. It can be verified, using the results given below, that this integrability condition is automatically satisfied. The solution is

$$c^{(1)} = \tilde{c}^{(1)}(x, y, t) + \partial_x \|c\| \int_{-h}^{z} \frac{dz'}{\kappa_3^{(0)}} \int_{-h}^{z'} (u^{(0)} - \|u^{(0)}\|)\, dz'' + \partial_y \|c\| \int_{-h}^{z} \frac{dz'}{\kappa_3^{(0)}} \int_{-h}^{z'} v^{(0)}\, dz'', \tag{7b}$$

the z-independent term $\tilde{c}^{(1)}$ being so chosen that $c^{(1)}$ has zero vertical average. At order δ^2 the forcing terms are more complicated. The integrability condition is that $\|c\|$ satisfies the evolution equation

$$\partial_t \|c\| + \|u^{(1)}\| \partial_x \|c\| + \|v^{(1)}\| \partial_y \|c\| + \partial_x \|(u^{(0)} - \|u^{(0)}\|)\, c^{(1)}\| + \partial_y (\|v^{(0)} c^{(1)}\| - \|\kappa_2^{(0)}\| \partial_y \|c\|) = 0, \tag{7c}$$

where $\|\ldots\|$ indicates a vertical average. To evaluate the coefficients in terms of $\|c\|$ we need to solve the equations of motion (6b–h).

To the first approximation the gradient Richardson number (6h) is zero and the eddy diffusivities $\kappa_i^{(0)}$ and $\nu_{ij}^{(0)}$ are those pertaining to homogeneous flows. Thus the basic longitudinal current is unaffected by the contaminant:

$$u^{(0)}(z) = G \int_{-h}^{z} \frac{(-z')}{\nu_{13}^{(0)}}\, dz', \quad \text{independent of } x \text{ and } y. \tag{8a}$$

The solution of the remaining leading-order equation, as obtained by Prych (1970), gives a description of the buoyancy-driven lateral secondary flow:

$$v^{(0)} = \partial_z \psi, \quad w^{(0)} = -\partial_y \psi, \tag{8b, c}$$

$$p^{(0)} = \alpha g \|c\| (z + \tfrac{1}{2}\tilde{\psi}) - G \int^{x} \mu(x', t)\, dx', \tag{8d}$$

$$\psi = \alpha g \partial_y \|c\| \int_{-h}^{z} dz' \int_{-h}^{z'} \left\{ \frac{z''^2 + z'' \tilde{\psi}}{2\nu_{23}^{(0)}} \right\} dz''. \tag{8e}$$

Here the constant $\bar{\psi}$ is so chosen that the stream function ψ is zero at the free surface. The correction μG to the y-independent pressure gradient arises because when there is vertical stratification (and reduced drag) the total flow along the channel can be maintained by a reduced longitudinal pressure gradient.

Since we are concerned with only weak stratification, it suffices to use a linear (Taylor's series) approximation for the change in the eddy viscosity. As is revealed in figure (2.14) of the Delft Hydraulics Laboratory literature survey (Breusers 1974), there is wide scatter in the available data. Here we use the formula (attributed to Vreugdenhil)

$$\nu_{13}^{(1)}/\nu_{13}^{(0)} = -\tfrac{10}{3}Ri^{(1)},$$

which is a reasonable approximation for $Ri^{(1)} < 0{\cdot}25$. Other empirical formulae, attributed in the Delft survey to Nelson and to Munk & Anderson, yield linear coefficients $-\frac{5}{2}$ and -5 respectively. Evaluating $Ri^{(1)}$ by means of the results (6h) and (7b), we find

$$\frac{\nu_{13}^{(1)}}{\nu_{13}} = -\frac{10}{3}\frac{\alpha g\partial_x\|c\|}{\kappa_3^{(0)}(\partial_z u^{(0)})^2}\int_{-h}^{z}(u^{(0)}-\|u^{(0)}\|)\,dz-\frac{10}{3}\frac{\alpha g\partial_y\|c\|}{\kappa_3^{(0)}(\partial_z u^{(0)})^2}\psi.$$

Proceeding to the order-δ terms in the longitudinal momentum equation (6b), we obtain

$$u^{(1)} = \alpha g\,\partial_x\|c\|U_1(z)+(\alpha g\,\partial_y\|c\|)^2\,U_2(z)+\alpha g\,\partial_y^2\,\|c\|U_3(z)+\mu u^{(0)}(z),$$

where $\mu(x,t)$ is so chosen that the perturbation longitudinal current $u^{(1)}$ makes no net contribution to the volume flux along the channel, and the functions $U_j(z)$ are defined as

$$U_1 = \int_{-h}^{z}\left\{\frac{z'^2+z'\bar{\psi}}{2\nu_{13}^{(0)}}\right\}dz'+\int_{-h}^{z}\frac{10}{3}\frac{dz'}{\kappa_3^{(0)}\partial_z u^{(0)}}\int_{-h}^{z'}(u^{(0)}-\|u^{(0)}\|)\,dz'', \tag{9a}$$

$$U_2 = \int_{-h}^{z}\frac{10}{3}\frac{(\psi/\alpha g\partial_y\|c\|)}{\kappa_3^{(0)}\partial_z u^{(0)}}\,dz', \tag{9b}$$

$$U_3 = \int_{-h}^{z}\frac{dz'}{\nu_{13}^{(0)}}\int_{z'}^{0}\partial_z u^{(0)}(\psi/\alpha g\partial_y\|c\|)\,dz''. \tag{9c}$$

We observe that U_1 can be associated with a combination of a longitudinal density-driven current and the reduction in turbulent intensity due to the part of the vertical stratification that is induced by the longitudinal density gradient, that U_2 is related to the corresponding transverse-induced turbulence reduction and that U_3 represents the vertically transported longitudinal momentum due to the secondary flow.

There is no need for us to pursue the further details of the second-order solution, because the vertically averaged velocity perturbations $\|u^{(1)}\|$ and $\|v^{(1)}\|$ are related via a stream function:

$$h\|u^{(1)}\| = \partial_y\Phi, \quad h\|v^{(1)}\| = -\partial_x\Phi. \tag{9d, e}$$

From the above expression for $u^{(1)}$ in terms of U_1, U_2 and U_3 we can derive a corresponding expression for μ and hence for Φ:

$$\Phi = \int_{y-}^{y}[\alpha g\partial_x\|c\|\,h\|U_1\|-\overline{\alpha g\partial_x\|c\|\,h\|U_1\|}]\,dy$$
$$+\int_{y-}^{y}[(\alpha g\partial_y\|c\|)^2h\|U_2\|-\overline{(\alpha g\partial_y\|c\|)^2h\|U_2\|}]\,dy+\alpha g\partial_y\|c\|\,h\|U_3\|. \tag{9f}$$

Here $y = y_-$ is the left-hand side of the channel (viewed from upstream) and the overbars denote cross-sectional average values. Without loss of generality we have chosen to set Φ equal to zero at the two side walls. For an infinite channel it suffices that we take $y_- = -\infty$ and neglect the averaged terms. The marked similarity between the first term in the expression (9*a*) for U_1 and the equation (8*e*) for ψ means that the contribution to $\|U_1\|$ from the longitudinal density-driven current is zero unless the eddy viscosities $\nu_{13}^{(0)}$ and $\nu_{23}^{(0)}$ for longitudinal and lateral flows have different vertical variation.

The inclusion of h factors and some integrations by parts with respect to z enable us to rewrite (7*c*):

$$\partial_t(h\|c\|) + \partial_y \Phi\, \partial_x \|c\| - \partial_x \Phi\, \partial_y \|c\| = \partial_x(h\,\|\partial_z c^{(1)} \int_{-h}^{z} (u^{(0)} - \|u^{(0)}\|)\, dz)$$

$$+ \partial_y(h\,\|\partial_z c^{(1)} \int_{-h}^{z} v^{(0)} dz\| + h\|\kappa_2^{(0)}\|\,\partial_y \|c\|).$$

Eliminating $\partial_z c^{(1)}$ and $v^{(0)}$ by means of (7*b*) and (8*b*), we are led to define the dispersion coefficients

$$E = \frac{1}{h}\int_{-h}^{0} \frac{dz}{\kappa_3^{(0)}} \left[\int_{-h}^{z} (u^{(0)} - \|u^{(0)}\|)\, dz'\right]^2, \tag{10a}$$

$$D_1 = \frac{1}{h}\int_{-h}^{0} dz \frac{\psi}{\kappa_3^{(0)}} \int_{-h}^{z} (u^{(0)} - \|u^{(0)}\|)\, dz', \tag{10b}$$

$$K = \|\kappa_2^{(0)}\| + \frac{1}{h}\int_{-h}^{0} \frac{\psi^2}{\kappa_3^{(0)}}\, dz. \tag{10c}$$

Thus, with respect to axes moving with the bulk velocity $\|u^{(0)}\|$, the contaminant dispersion equation takes the form (1):

$$\partial_t(h\|c\|) + \partial_y \Phi \partial_x \|c\| - \partial_x \Phi \partial_y \|c\| = (\partial_x, \partial_y) \begin{bmatrix} E & D_1 \\ D_1 & K \end{bmatrix} \begin{bmatrix} h\,\partial_x \|c\| \\ h\,\partial_y \|c\| \end{bmatrix}. \tag{1}$$

The formulae for the longitudinal and lateral dispersion coefficients E and K agree with the work of Elder (1959) and of Prych (1970) respectively. Moreover, as was noted in the introduction, (1) has the correct limiting form when the dispersion is predominantly along or across the flow. Although the present analysis is restricted to channels of constant depth, the results (9*f*) and (1) indicate the way in which variations in depth can be expected to modify the equations.

4. Logarithmic velocity profile

If we are to obtain quantitative results then it is necessary to specify the vertical structure of the eddy diffusivities $\nu_{13}^{(0)}$, $\nu_{23}^{(0)}$ and $\kappa_3^{(0)}$, and to determine the many coefficients involved in (9*f*) and (1). The simplest possibility, as considered by Prych (1970), is to use a depth-averaged (quasi-laminar) value of the diffusivities. Here we follow Elder (1959), and use a more realistic model which corresponds to the basic velocity profile being logarithmic. In this model the diffusivity distributions for the vertical transports of momentum and concentration are parabolic:

$$\nu_{13}^{(0)} = u_* h \nu_{13}'(1-\eta)\,\eta, \quad \text{where} \quad \eta = (h+z)/h, \tag{11a}$$

with similar formulae for $\nu_{23}^{(0)}$ and $\kappa_3^{(0)}$. Here ν'_{13} is a dimensionless constant and u_* is the friction velocity. Later in this section we invoke a stronger form of the Reynolds hypothesis and take the diffusivities $\nu_{13}^{(0)}$, $\nu_{32}^{(0)}$ and $\kappa_3^{(0)}$ all to be equal.

We notice that the diffusivity tends to zero as the channel bottom is approached. If this were strictly the case then singularities would arise owing to the occurrence of diffusivities in the denominators of many of the above integrals. A correct method to resolve the singularities is to recognize that there is a thin laminar sublayer in which the eddy diffusivities remain non-zero. In the context of present study, it can be shown that the results are insensitive to the details of the sublayer. Thus it suffices to take an algebraically convenient model. Here we choose to apply the bottom boundary conditions at the displaced position $\eta = \eta_*$, where

$$\eta_* = \exp\{-[\nu'_{13}\,\|u^{(0)}\|/u_* + 1](1-\eta_*)\}. \tag{11b}$$

This definition ensures that $\|u^{(0)}\|$ is equal to the vertical average of $u^{(0)}$ over the slightly reduced region $\eta_* < \eta < 1$. We observe that η_* is exponentially small with respect to that small parameter $u_*/\|u^{(0)}\|$. Thus we shall neglect powers of η_*. The fact that the final results do not depend upon η_* is related to the unimportance of the details of the sublayer.

By definition, the friction velocity u_* is related to the pressure gradient G:

$$G = u_*^2/h.$$

Thus equation (8a) for the undisturbed longitudinal velocity yields the anticipated logarithmic profile

$$\begin{aligned} u^{(0)} &= (u_*/\nu'_{13})\ln(\eta/\eta_*) \\ &= \|u^{(0)}\| + (u_*/\nu'_{13})(1+\ln\eta) + O(\eta_*). \end{aligned} \tag{12a}$$

The other major formula (8e) for the leading-order flow becomes

$$\psi = \frac{1}{4}\frac{\alpha g \partial_y \|c\| h^3}{\nu'_{23} u_*}\left[\eta - \eta^2 + \frac{u_*}{\nu'_{13}\|u^{(0)}\|}\eta\ln\eta\right], \tag{12b}$$

where in accord with the above prescription further terms involving powers of η_* have been neglected.

For future reference we note that the gradient Richardson number, as defined by (6h), is given to the first approximation by

$$Ri = \alpha g \partial_x \|c\| \frac{h^2 \nu'_{13}}{u_*^2 \kappa'_3}\frac{\eta^2 \ln\eta}{1-\eta} + \tfrac{1}{4}(\alpha g \partial_y \|c\|)^2 \frac{h^4 (\nu'_{13})^2}{u_*^4 \kappa'_3 \nu'_{23}}\left[\eta^2 + \frac{u_*}{\nu'_{13}\|u^{(0)}\|}\frac{\eta^2\ln\eta}{1-\eta}\right]. \tag{12c}$$

This expression enables us to check whether the vertical stratification is consistent with the use of a middle-field model (e.g. $Ri < 0{\cdot}25$).

Using these results (12a, b) in the integrals (9a, b, c) for the horizontal circulation coefficients U_j, we obtain

$$\|U_1\| = -\frac{5}{6}\frac{h^2}{u_* \kappa'_3}, \quad \|U_2\| = \frac{5}{18}\frac{h^4 \nu'_{13}}{u_*^3 \kappa'_3 \nu'_{23}}\left[1 + \frac{3u_*}{2\nu'_{13}\|u^{(0)}\|}\right], \tag{13a, b}$$

$$\|U_3\| = \frac{h^3 \|u^{(0)}\|}{8\nu'_{13}\nu'_{23}u_*^2}\left[1 - \frac{5}{2}\frac{u_*}{\nu'_{13}\|u^{(0)}\|} - 2\left(\frac{u_*}{\nu'_{13}\|u^{(0)}\|}\right)^2\right]. \tag{13c}$$

Similarly, for the dispersion terms we find

$$E = 0{\cdot}4041\, hu_*/(\nu'_{13})^2 \kappa'_3, \tag{14a}$$

$$D_1 = \frac{-\alpha g\,\partial_y \|c\| h^3}{16 u_* \nu'_{13}\nu'_{23}\kappa'_3}\left[1 - \frac{1{\cdot}6164\, u_*}{\nu'_{13}\|u^{(0)}\|}\right], \tag{14b}$$

$$D_2 = \frac{h^5}{96\kappa'_3(\nu'_{23})^2 u_*^3}\left[1 - \frac{3u_*}{\nu'_{13}\|u^{(0)}\|} + 2{\cdot}4246\left(\frac{u_*}{\nu'_{13}\|u^{(0)}\|}\right)^2\right]. \tag{14c}$$

Conveniently, the only non-elementary integral is the one evaluated by Elder (1959) and it is this that gives rise to the decimal terms.

To complete our determination of the coefficients in the dispersion equation we need to know κ'_3, ν'_{13}, ν'_{23} and $\|\kappa_2^{(0)}\|$. For a variety of experimental results (Fischer 1973), a reasonable specification is

$$\kappa'_3 = \nu'_{13} = \nu'_{23} = 0{\cdot}4, \quad \|\kappa_2^{(0)}\| = 0{\cdot}15 hu_*. \tag{15}$$

The slight uncertainty in these values means that it is adequate to retain only the leading term in (13) and (14). Thus we arrive at the results for the coefficients (2*a*–*d*) quoted in the introduction to this paper:

$$E = 6{\cdot}3hu_*, \quad D_1 = 0{\cdot}98h^3\alpha g\,\partial_y\|c\|/u_*,$$

$$K = 0{\cdot}15hu_* + 0{\cdot}16h^5(\alpha g\,\partial_y\|c\|)^2/u_*^3,$$

$$\Phi = -2{\cdot}1\int_{y-}^{y}\left[\frac{h^3}{u_*}\alpha g\,\partial_x\|c\| - \overline{\frac{h^3}{u_*}\alpha g\,\partial_x\|c\|}\right]dy$$

$$+\,0{\cdot}70\int_{y-}^{y}\left[\frac{h^5}{u_*^3}(\alpha g\,\partial_y\|c\|)^2 - \overline{\frac{h^5}{u_*^3}(\alpha g\,\partial_y\|c\|)^2}\right]dy + 0{\cdot}78\,h^4\alpha g\,\partial_y\|c\|\,\|u^{(0)}\|/u_*^2.$$

The numerical factor 6·3 in the longitudinal dispersion term is slightly greater than the value 5·9 given by Elder (1959), owing to his using the larger value 0·41 for the von Kármán constant. On the other hand Prych (1970) uses the smaller value 0·38, but his result 0·052 for the nonlinear lateral dispersion coefficient is considerably reduced from the present estimate 0·16. Indeed, for the same specification of the turbulence the ratio of the D_2 coefficients for logarithmic and parabolic velocity profiles is 70:19.

The explanation for this considerable change lies in the repeated occurrence of the diffusivities in the denominators, which gives emphasis to those regions of the fluid with low diffusivities. Experiments performed by Ippen, Harleman & Lin (1960) with an externally imposed uniform turbulence field gave dispersion rates only marginally less than Prych's theoretical results. However, in Prych's own experiments there was no external turbulence source and his figures (4.3*a*, *b*) reveal that the velocity profile was logarithmic to a high degree of accuracy. Thus, with the advantage of hindsight, we can explain why his theory gave considerable underestimates of the dispersion.

5. Simplified dispersion equation

The single equation (1) in two spatial dimensions is a considerable simplification from the eight equations (4a–h) in three spatial dimensions. Furthermore, with respect to the middle field the loss of generality has been minimal. However, the model equation is more complicated than we might have anticipated. The reason for this is that not only are there terms corresponding to each physical effect, but there are also some combination terms. For example, the term most naturally associated with the vertical shear of the primary current is the longitudinal dispersion coefficient E. Yet (9a) and (10b) reveal that $u^{(0)} - \|u^{(0)}\|$ is also involved in the two further terms $\|U_1\|$ and D_1.

After a sufficiently long time the density gradients become negligible and the contaminant distribution evolves according to the linear diffusion equation

$$\partial_t(h\|c\|) = \partial_x(hE\,\partial_x\|c\|) + \partial_y(hD_0\,\partial_y\|c\|).$$

For practical purposes it often suffices to know how rapidly the nonlinearity vanishes and how the eventual linear solution differs from a global linear solution (e.g. the excess variance due to buoyancy). Because the contaminant distribution is continually widening, the largest influences upon the asymptotic solution are due to the last significant nonlinear effects. One means of addressing this problem is to seek nonlinear corrections to the final approach to normality (Barton 1976; Smith 1978). In this section we seek instead a simplified dispersion equation which retains the last nonlinear effects (on the premise that the nonlinearity persists into the middle field). To do this we interpret $u_*/\|u^{(0)}\|$ as a small parameter, and ascertain the maximum-generality scalings. For Prych's (1970) experiments the range of values for this parameter is from 0·047 to 0·10 depending upon the flow conditions.

As in § 2, we take H and $\|u\|$ to be the basic dimensional quantities. Suppose that the reduced gravity and the longitudinal and transverse gradients have the following sizes:

$$\alpha g \sim \frac{\|u\|^2}{H}\left(\frac{u_*}{\|u\|}\right)^{-G}, \quad \partial_x \sim \frac{1}{H}\left(\frac{u_*}{\|u\|}\right)^{L}, \quad \partial_y \sim \frac{1}{H}\left(\frac{u_*}{\|u\|}\right)^{W}$$

where the exponents G, L and W connote gravity, length and width. Thus the exponents of the U_1, U_2, U_3, E, D_1, D_0 and D_2 terms in (1) are

$$2L-G-1, \quad L+2W-2G-3, \quad L+2W-G-2,$$

$$2L-N+1, \quad L+2W-G-1, \quad 2W+1, \quad 4W-2G-3,$$

respectively. Here the extraneous exponent N allows for the fact that the numerical factor 6·3 in the longitudinal dispersion coefficient (2a) is more than forty times as large as the numerical factor 0·15 in the lateral turbulent diffusion coefficient (15). The final simplified model equation will include those terms with the lowest exponents (i.e. the formally largest terms).

By hypothesis, the lateral turbulence and secondary flow terms must be included [i.e. the features of Prych's middle-field equation (3)]. This implies that

$$G = W - 2$$

and the relative exponents of the respective terms in (1) can be written as

$$2L-3W,\quad L-2W,\quad L-W-1,$$
$$2L-2W-N,\quad L-W,\quad 0,\quad 0.$$

The second, third and fourth exponents are zero provided that we choose

$$G=-1,\quad L=2,\quad W=1,\quad N=2,$$

and the remaining exponents are positive. The importance of these scalings is that, subject to $u_*/\|u\|$ being small, they lead to the retention of the maximum number of physical effects.

The resulting dispersion equation, in axes moving at the bulk velocity $\|u^{(0)}\|$, is of the form

$$\partial_t(h\|c\|)+\partial_y\Phi\partial_x\|c\|-\partial_x\Phi\partial_y\|c\| = \partial_x(hE\partial_x\|c\|)+\partial_y(h[D_0+(\alpha g\partial_y\|c\|)^2D_2]\partial_y\|c\|) \quad (16a)$$

with

$$\Phi=\int_{y-}^{y}[(\alpha g\partial_y\|c\|)^2h\|U_2\|-\overline{(\alpha g\partial_y\|c\|)^2h\|U_2\|}]\,dy+\alpha g\partial_y\|c\|h\|U_3\|. \quad (16b)$$

The expressions for the coefficients E, D_0, D_2, $\|U_2\|$ and $\|U_3\|$ can be obtained by comparison with the full equations (1) and (2*a–d*). It happens that the simplifications have led to the deletion of the interaction terms noted in the first paragraph of this section. Thus there is a one-to-one correspondence between the major physical effects and the coefficients in (16*a*, *b*). E is associated with the vertical shear of the longitudinal current, D_0 with the lateral turbulent diffusion, D_2 with the vertical shear of the buoyancy-driven secondary flow, $\|U_2\|$ with the turbulent drag reduction due to the weak vertical stratification, and $\|U_3\|$ with the redistribution of longitudinal momentum by the secondary flow.

Using the above scalings, we find that the estimate (12*c*) of the gradient Richardson number can be simplified to

$$Ri=\frac{(\alpha g\partial_y\|c\|)^2h^4}{4u_*^4}\left(\frac{\nu_{13}'^2}{\kappa_3'\nu_{23}'}\right)\eta^2,$$

where η ranges between 0 and 1.

6. Comparison between theory and experiment

Intuitive arguments, as presented in the introduction, suggest that for a light contaminant (i.e. α positive) each of the terms in the dispersion equation (1) tends to increase the rate of spreading. Here, for the simplified equation (16), we give analytic confirmation of those conclusions. The specific case which we study is a steady plume in an extremely wide constant-depth channel. Thus we can test these theoretical predictions against Prych's experimental results.

The method used has its basis in the recent work of the author (Smith 1978). There it is shown that if a contaminant distribution is represented in terms of its cumulants $a_n(x)$, viz.

$$\|c\|=\frac{Q}{\alpha gh\|u^{(0)}\|}\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}\left\{1+\sum_{n=1}^{\infty}\frac{a_n}{n!}\mathrm{He}_n(y/\sigma)\right\}\exp\{-\tfrac{1}{2}(y/\sigma)^2\},$$

then the He_2 coefficient of the dispersion equation yields an evolution equation for the variance $\sigma^2(x)$. Here the source strength Q is the rate at which weight deficit (or excess) is being discharged into the stream and the He_n are Hermite polynomials defined by

$$He_n(Y)\exp(-\tfrac{1}{2}Y^2) = (-d/dY)^n \exp(-\tfrac{1}{2}Y^2).$$

The major mathematical difficulty is to calculate the coefficients $a_n(x)$ (Smith 1978). However, we can infer that far downstream the cumulants tend to zero, and therefore a reasonable approximation to the equation for σ^2 can be obtained by neglecting all the cumulants. In this Gaussian approximation we can explicitly evaluate the integral and derivative terms in (16), and it is straightforward to determine the corresponding He_2 coefficients.

With respect to stationary axes, the resulting equation for the variance $\sigma^2(x)$ is

$$E\frac{d^2\sigma^2}{dx^2} + \left[\|u^{(0)}\| - \frac{Q^2\|U_2\|}{2\pi 3^{\frac{1}{2}}h^2\|u^{(0)}\|^2\sigma^4} - \frac{Q\|U_3\|}{\pi^{\frac{1}{2}}16h\|u^{(0)}\|\sigma^3}\right]\frac{d\sigma^2}{dx} = \left[2D_0 + \frac{Q^2D_2}{\pi 3^{\frac{3}{2}}h\|u^{(0)}\|\sigma^4}\right]. \quad (17)$$

Since the coefficients D_2, $\|U_2\|$ and $\|U_3\|$ are all positive [see (14*c*) and (13*b*, *c*)], it follows that for light contaminants (i.e. with Q positive) the secondary flow, the turbulence reduction and the redistribution of longitudinal momentum all enhance the dispersion. On changing the sign of Q we find that the $\|U_3\|$ term is reversed and tends to reduce the dispersion. Thus, in qualitative agreement with Prych's (1970) experimental findings, we predict that dense contaminants are dispersed more slowly than are light contaminants.

Following Prych (1970), we can make our results non-dimensional by introducing the dimensionless variance V, distance downstream X, source width B and source strength S:

$$V = \frac{\sigma^2}{h^2}, \quad X = \frac{xD_0}{h^2\|u^{(0)}\|}, \quad B = \frac{b}{h}, \quad S = \frac{|Q|h}{D_0^{\frac{3}{2}}\|u^{(0)}\|}.$$

A useful by-product of non-dimensionalization is that it leads to a standard form for the variance equation (17). If, as in Prych's experiments, there are a number of different flow conditions then a standard form permits us to define a meaningful 'averaged' equation. Using (13*b*, *c*), (14*a*, *c*) and Prych's results for the flow and turbulence properties, we obtain the averaged coefficients

$$0{\cdot}00586\frac{d^2V}{dX^2} + \left[1 - \frac{1{\cdot}59\times 10^{-6}S^2}{V^2} \pm \frac{5{\cdot}82\times 10^{-4}}{V^{\frac{3}{2}}}S\right]\frac{dV}{dX} = 2 + \frac{3{\cdot}16\times 10^5}{V^2}S^2. \quad (18)$$

Here the choice of sign for the momentum redistribution term depends upon whether the contaminant is dense or light.

The corresponding formula for the zero moment of the gradient Richardson number across the plume is

$$Ri_{av} = 6{\cdot}02\times 10^{-7}S^2/V^2. \quad (19)$$

Here the averaging is vertically, laterally and with respect to the different flow conditions. By comparing the sizes of terms in the variance equation (18) with the expression (19), we see that the contaminant dispersion can be extremely nonlinear before the averaged Richardson number becomes too large for the consistency of a middle-field model (i.e. a model based upon the assumption that there is nearly complete vertical mixing).

A phase-plane analysis of (18) reveals that while the dV/dX coefficient is negative there is extremely rapid growth of the variance. Once this coefficient has changed sign the solution closely follows that of the first-order equation, in which the d^2V/dX^2 term is neglected. Thus a simple and accurate means of approximating the solutions of (16) is via the implicit solution of the associated first-order equation:

$$X = \int_{V_0}^{V} \left[1 - \frac{1{\cdot}59 \times 10^{-6} S^2}{V^2} \pm \frac{5{\cdot}82 \times 10^{-4} S}{V^{\frac{3}{2}}}\right] \left[2 + \frac{3{\cdot}16 \times 10^{-5} S^2}{V^2}\right]^{-1} dV. \qquad (20a)$$

Here the starting position for the integral is the larger of the actual variance at the discharge $\frac{1}{12}B^2$ and the unique positive root of the equation

$$1 - 1{\cdot}59 \times 10^{-6} S^2 V^{-2} \pm 5{\cdot}82 \times 10^{-4} S V^{-\frac{3}{2}} = 0 \qquad (20b)$$

(i.e. the sign change for the dV/dX coefficient). This procedure implicitly assumes that any stratification-dominated near field does not extend far downstream and does not greatly contribute to the variance.

For large values of V the result (20a) asymptotes to

$$X \sim \tfrac{1}{2}(V - V_0) - \int_{V_0}^{\infty} \left[\frac{1{\cdot}74 \times 10^{-5} S^2}{V^2} \mp \frac{5{\cdot}82 \times 10^{-4} S}{V^{\frac{3}{2}}}\right] \left[2 + \frac{3{\cdot}16 \times 10^{-5}}{V^2}\right]^{-1} dV,$$

while the solution for a neutrally buoyant discharge is

$$X = \tfrac{1}{2}V - \tfrac{1}{24}B^2.$$

At a large distance downstream of the outlet the difference between the alternative results for the variance is

$$(V_0 - \tfrac{1}{12}B^2) + \int_{V_0}^{\infty} \left[\frac{1{\cdot}74 \times 10^{-5} S^2}{V^2} \mp \frac{5{\cdot}82 \times 10^{-4}}{V^{\frac{3}{2}}}\right] \left[1 + \frac{1{\cdot}58 \times 10^{-5}}{V^2}\right]^{-1} dV. \qquad (21)$$

Figure 6 compares this theoretical prediction for the excess variance due to buoyancy with Prych's (1970) experimental results for dense contaminants (i.e. with the minus sign).

Prych's experiments extend to strongly stratified cases (see the vertical concentration profiles in his figures 4.5 and 4.6). A generous upper bound for the applicability of the present theory is that the averaged Richardson number should be less than 0·25 at the discharge. For the four dimensionless source widths 0·1, 3·08, 4·94, 27·7 the respective upper bounds for the dimensionless source strength are

0·53, 510, 1300, 41000.

Outside these bounds the theory underestimates the dispersion by as much as 40%. For wider, less buoyant plumes there is good agreement between theory and experiment.

The relative sizes of the terms in (21) permit us to ascertain which of the three retained buoyancy effects is dominant. Appropriately, many of Prych's experiments concern situations in which secondary flow is the main mechanism. Thus, using the Erdogan–Chatwin equation (3) with the value of D_2 derived in the present paper, the author (Smith 1978) has obtained results which closely resemble figure 6. The additional physical effects included here reduce the dispersion for small S and slightly increase the dispersion for large S.

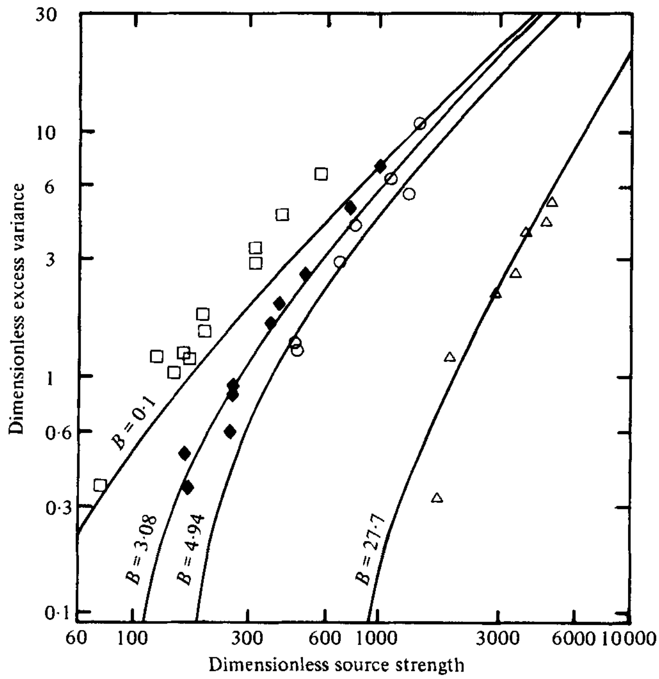

FIGURE 6. Comparison between the present theoretical and Prych's (1970) experimental results.

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